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Gauge theory renormalizations from the open bosonic string

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Abstract

We present a unified point of view on the different methods available in the literature to extract gauge theory renormalization constants from the low-energy limit of string theory. The Bern-Kosower method, based on an off-shell continuation of string theory amplitudes, and the construction of low-energy string theory effective actions for gauge particles, can both be understood in terms of strings interacting with background gauge fields, and thus reproduce, in the low-energy limit, the field theory results of the background field method. We present in particular a consistent off-shell continuation of the one-loop gluon amplitudes in the open bosonic string that reproduces exactly the results of the background field method in the Feynman gauge.

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1 Introduction

The interest in extracting from string theory information about the effective interactions of the low-energy degrees of freedom is as old as string theory itself.

Using a variety of approaches several authors [1]-[6] have studied how to extract one-loop field theoretical quantities, such as scattering amplitudes, renormalization constants and β functions, from ultraviolet finite string theories.

In particular, it has become clear through the work of Bern and Kosower that, whether or not string theory is a fundamental theory of nature, it is a valuable tool to obtain results in quantum field theory that would have been very hard to get otherwise. They extracted from string theory the QCD β function [4], and, more generally, they used string theory to generate rules to calculate one-loop amplitudes in QCD [6]. Using these rules, they were able to calculate all the helicity amplitudes involving five gluons at one loop [7], a remarkable achievement from the point of view of traditional techniques. Similar methods were also applied to gravity [8].

Other approaches are based on the study of strings interacting with external backgrounds [3, 9]. With this method one can compute from string theory the running of the low-energy couplings, determine threshold corrections in string-inspired grand unified theories [3], and obtain several other low-energy quantities of interest [10].

The main obstacle to the extraction of this kind of information lies in the fact that present day string technology is essentially restricted to the calculation of on-shell quantities, whereas renormalization in field theory is best carried out for general, off-shell Green's functions.

This obstacle can be circumvented, however, either by attempting an off-shell continuation of the string theory amplitudes calculated with traditional techniques [5], or by evaluating low-energy effective actions in the presence of background fields satisfying the equations of motion, as done in Ref. [3].

The purpose of this letter is to establish a connection between the two methods, and to show how they are both strictly related, in the field theory limit, to the background field method. In the process, we believe we will clarify several issues concerning the relationship between string theory amplitudes and renormalizations, and we will show the consistency, from the point of view of the gauge theory, of a particular prescription for the off-shell continuation of the string amplitudes, different from the one given in Ref. [5].

We must emphasize that we are not going to require the overall consistency of the string theory we work with. Rather, we consider string theory as an efficient bookkeeping device that may generate economically field theoretical results, in the spirit of Bern and Kosower's approach. Thus we work with the open bosonic string, and we formally continue the consistent on-shell amplitudes off shell, and to arbitrary space-time dimension. We will see at the end that, having taken correctly the field theory limit, and decoupling the $1/\alpha'$ tachyonic divergences, we will recover consistent field theory results.

We will briefly discuss how gauge theory renormalizations are related to string theoretical amplitudes and effective actions. We will then consider in some detail the two-gluon amplitude in the open bosonic string, restricting our attention to the $SU(N)$ gauge group, and we will show how, in the field theory limit, it coincides precisely with the amplitude calculated in the background field Feynman gauge, off shell and with dimensional regularization. We will argue that the prescription used to extend off shell the two-gluon amplitude can be applied to other amplitudes as well, and it gives consistent renormalizations for the gauge theory, obeying the appropriate Ward identities. We will then examine the one-loop effective action for the open bosonic string, in a slowly varying Yang-Mills background, and we will show that it can be interpreted in practice as an infrared limit of the previously calculated two-gluon amplitude. Also in this case we can establish a precise connection with the corresponding calculation in field theory, which is again carried out with the background field method, but this time regularized with a proper time cutoff.

To lay the grounds for an extension of the present calculations to higher loops, we will use from the beginning the h -loop operator formalism discussed in Ref. [11], but the reader is advised that only one-loop calculations will be performed, and it is straightforward to map the present notation into the more standard one of, say, Ref. [12]. Several technical points and the details of the calculations are left to a subsequent publication [13].

2 A note on renormalization constants

String theory tells us how to compute on-shell scattering amplitudes among physical string states. In field theory, on the other hand, we know through the reduction formulas that on-shell scattering amplitudes are given by the on-shell limit of the truncated Green's functions of the theory, multiplied by the appropriate powers of the residues of the propagators.

When we calculate one-loop string theory amplitudes, and take the field theory limit, generating the ultraviolet divergences of the low-energy effective theory, we are thus calculating *unrenormalized*, on-shell, truncated Green's functions.

This is a problem if the low-energy theory contains gauge bosons, with the associated mass (infrared and collinear) divergences, because these Green's functions are ill-defined even in field theory, due to the possible cancellation of mass and ultraviolet divergences. This is apparent if one uses dimensional regularization: in this case the Green's function receives contributions from diagrams in which the loop is isolated on an external leg, and since such a loop integral is defined to vanish in dimensional regularization, the Green's function suffers, even *in field theory*, from a $0/0$ ambiguity.

It is easy to deal with this ambiguity in field theory: one simply calculates the Green's function off shell, thus getting rid of mass divergences, and subsequently

renormalizes it, subtracting out the ultraviolet divergences that are left. Finally, one takes the on-shell limit, and recovers a purely infrared divergence, which is a genuine contribution to the S -matrix, and will be cancelled when forming a sufficiently inclusive cross section.

In string theory the situation is complicated by the fact that there is no obvious consistent way of continuing the scattering amplitudes off shell, thus disentangling mass from ultraviolet divergences in the field theory limit. The amplitudes at one loop are inherently ambiguous, so one has to choose a prescription to regularize this ambiguity, and only a posteriori one can verify the consistency of the calculation [5].

Assuming that one such prescription has been chosen, the computation of renormalization constants is relatively straightforward; one just has to keep in mind the relationship between truncated and one-particle irreducible Green's functions (proper vertices). To set our notations, consider a pure gauge theory, denote by $\Gamma_M(p_1, \dots, p_{M-1})$ the proper M -point vertex, and by $W_M(p_1, \dots, p_{M-1})$ the truncated connected M -point Green's function, with momentum conservation already taken into account. Let Z_A , Z_3 and Z_4 be the multiplicative renormalization constants associated with the gluon wave function and with the three- and four-point vertices respectively. Then the relation between the first few truncated functions and the corresponding proper vertices is

$$\begin{aligned} W_2(p) &= -i(\Gamma_2(p))^{-1} \left(\Gamma_2^{(0)}(p) \right)^2 , \\ W_3(p_1, p_2) &= i\Gamma_3(p_1, p_2) \prod_{k=1}^3 \left[(\Gamma_2(p_k))^{-1} \Gamma_2^{(0)}(p_k) \right] , \\ W_4(p_1, p_2, p_3) &= i \left[\Gamma_4(p_1, p_2, p_3) - \sum_{(i,j)} \Gamma_3(p_i, p_j) (\Gamma_2(p_i + p_j))^{-1} \Gamma_3(p_k, p_l) \right] \\ &\quad \prod_{k=1}^4 \left[(\Gamma_2(p_k))^{-1} \Gamma_2^{(0)}(p_k) \right] , \end{aligned} \tag{1}$$

where $i \left(\Gamma_2^{(0)}(p) \right)^{-1}$ is just the free propagator, and the sum in the last equation is over the three channels of the four point amplitude. As a consequence, the truncated Green's functions must be renormalized according to

$$\begin{aligned} W_2 &= W_2^{(R)} Z_A , \\ W_3 &= W_3^{(R)} Z_3^{-1} Z_A^3 , \\ W_4 &= W_4^{(R)} Z_4^{-1} Z_A^4 , \end{aligned} \tag{2}$$

where the Ward identity $Z_4 = Z_3^2 Z_A^{-1}$ has been used.

The ultraviolet divergences encountered in the field theory limit of a string amplitude are thus related to the combinations of renormalization constants given by Eq. (2). Having established this, we go on to the calculation of the two-gluon amplitude, using the open bosonic string.

3 The two-gluon amplitude

To compute the M -gluon, h -loop amplitude $A_M^{(h)}$ in the open bosonic string in d dimensions, one can use the operator formalism discussed in Ref. [11], *i.e.* one can saturate the M -point h -loop vertex $V_{M;h}$ with gluon states of momenta p_i^μ , polarizations ε_i^μ , and color indices a_i . The field theory limit of $A_M^{(h)}$ yields the h -loop contribution to the on-shell, connected, truncated Green's functions W_M discussed in the previous section.

For $M = 2$, it is apparent that the result is ill-defined on shell, since all the kinematical invariants containing the gluon momenta vanish on shell. Furthermore, an ambiguity persists for any M [5], and it is twofold: on the one hand, if the external states are kept on the mass shell, the field theory limit contains ratios of vanishing invariants, and is thus ambiguous, not unlike the unrenormalized on-shell truncated Green's functions W_M . On the other hand, if one attempts a naïve continuation off shell, the world-sheet projective invariance of the amplitude is broken, and the result acquires a spurious dependence on the local coordinates of the punctures. In fact, the M -point vertex $V_{M;h}$ depends on the Koba-Nielsen variables z_i through M projective transformations $V_i(z)$, satisfying the constraint $V_i(z_i) = 0$. The on-shell amplitude is invariant under changes in the V_i 's, but the off-shell continuation is not. This is again not unlike the situation encountered in the gauge theory, where the on-shell limit is gauge invariant, but the off-shell continuation is not.

This twofold ambiguity was solved in [5] with a particular choice of the V_i 's, leading to non-ambiguous results in almost all cases. That choice turns out to be equivalent to the request that the gluon wave-function renormalization vanish.

Here we propose a different solution of the ambiguity, and we show that, with our prescription, the field theory limit is consistent and unambiguous, and it corresponds to the background field Feynman gauge. Our prescription is simply to treat the freedom to choose the V_i 's as a gauge freedom, which we fix to achieve the maximum simplification in the calculations. Our choice, at one loop, is

$$V_i(z) = z_i(z - z_i) \quad , \tag{3}$$

while at h loops it may be necessary to choose V_i 's depending on the moduli of the surface. Next we continue the momenta off the mass shell with the simplest choice preserving Bose symmetry, namely we set

$$p_i^2 = m^2 \quad , \quad \forall i \quad , \tag{4}$$

while we keep the transversality condition $\varepsilon_i \cdot p_i = 0$ to simplify the calculation. Relaxing this condition would generate more cumbersome expressions but would not change our results. In string theory, the choice of Eq. (4) is suggested by the observation [14] that it is possible to construct consistent string theories in which the entire spectrum is shifted by a fixed amount m^2 , by considering curved space-times,

and in particular certain coset spaces. Since we are interested in the field theory limit, and specifically in ultraviolet-dominated quantities, such as renormalization constants, we can simply take the curvature of space-time as an infrared regulator, and we need not worry about global properties.

Having established our prescription, let us start by writing down the two-gluon, h -loop correlation function derived using the operator formalism. It is

$$A_2^{(h)} = C_h \mathcal{N}_0^2 \int [dm]_h \exp [p_1 \cdot p_2 \mathcal{G}(z_1, z_2)] [V'_1(0) V'_2(0)]^{-(p_1 \cdot p_2)/2} \\ [\varepsilon_1 \cdot \varepsilon_2 \partial_{z_1} \partial_{z_2} \mathcal{G}(z_1, z_2) + \varepsilon_1 \cdot p_2 \varepsilon_2 \cdot p_1 \partial_{z_1} \mathcal{G}(z_1, z_2) \partial_{z_2} \mathcal{G}(z_1, z_2)] , \quad (5)$$

where z_1 and z_2 are the real Koba-Nielsen variables associated with the two gluons, $[dm]_h$ is the h -loop measure on moduli space for the open bosonic string [11], and $\mathcal{G}(z_1, z_2)$ is the world-sheet bosonic Green's function at h loops. C_h is the h -loop vertex normalization factor containing the d -dimensional Yang-Mills coupling constant g_d , the inverse string tension α' , and the appropriate $SU(N)$ Chan-Paton factor, while \mathcal{N}_0 is the normalization of the gluon state. We have omitted a momentum conservation delta function, and we are measuring momenta in units of $(2\alpha')^{-1/2}$. The amplitude $A_2^{(h)}$ corresponds to a planar string diagram with h loops, and two punctures on one boundary.

Eq. (5) is a remarkable expression, as any field theorist will realize. It generates, in the field theory limit, all the loop corrections to the Yang-Mills two-point function, something for which no closed expression exists in field theory. In particular, since we will show that in the field theory limit one recovers the background field method, Eq. (5) is a master formula containing all the information necessary to determine the multi-loop Yang-Mills β function.

Specializing to one loop ($h = 1$), the measure on moduli space will depend on five variables: the two Koba-Nielsen variables z_1 and z_2 , and the three moduli of the projective transformation that defines the annulus, which can be chosen (in the Schottky parametrization) as the multiplier k and the two fixed points η and ξ [11]. Three of these variables can be fixed using projective invariance. Our choice is to fix $\eta = 0$, $z_1 = 1$, and $\xi = \infty$. Then the integration region for the remaining moduli z_2 and k can be determined by going back to the sewing procedure used to construct the one-loop two-point vertex from the tree-level four-point one [15]. It is given by

$$1 \geq z_2 \geq k \geq 0 . \quad (6)$$

With these choices, the modular measure at one loop becomes,

$$[dm]_1 = dz_2 dk \frac{1}{k^2} \left(-\frac{1}{2\pi} \log k \right)^{-d/2} \prod_{n=1}^{\infty} (1 - k^n)^{2-d} . \quad (7)$$

Notice that the multiplier k is related to the usual modular parameter $\tilde{\tau}$ by

$$k = e^{2\pi i \tilde{\tau}} . \quad (8)$$

Below, we will use instead of $\tilde{\tau}$, which is purely imaginary, the real variable $\tau = -i\pi\tilde{\tau}$, which is integrated between 0 and ∞ . Similarly, instead of the Koba-Nielsen variables z_i , we will use the real variables $\nu_i = -\frac{1}{2}\log z_i$, so that with our choices $\nu_1 = 0$ while ν_2 is integrated between 0 and τ [16].

The bosonic Green's function $\mathcal{G}(z_1, z_2)$ at one loop is given by

$$\mathcal{G}(z_1, z_2) = \log \left[-2\pi i \frac{\theta_1\left(\frac{i}{\pi}(\nu_2 - \nu_1) \mid \frac{i}{\pi}\tau\right)}{\theta'_1\left(0 \mid \frac{i}{\pi}\tau\right)} \right] - \frac{(\nu_2 - \nu_1)^2}{\tau} - \nu_1 - \nu_2 , \quad (9)$$

where θ_1 is the Jacobi θ -function. Eq. (9) is related to the more commonly used one-loop correlation function $G(\nu)$ of, say, Ref. [12], by

$$G(\nu_2 - \nu_1) = \mathcal{G}(z_1(\nu_1), z_2(\nu_2)) + \nu_1 + \nu_2 . \quad (10)$$

Our choice of the V_i 's, Eq. (3), is made precisely to cancel the factors arising from $\nu_1 + \nu_2$ in the exponent of Eq. (5).

We are now ready to write down explicitly the one-loop version of Eq. (5). Following the strategy of Ref. [6], we first integrate by parts the term containing a double derivative of the Green's function. Then we mimick dimensional regularization by formally taking the dimension of space-time to be $d = 4 - 2\epsilon$. Finally we write down explicitly all the numerical factors, and reinstate α' in preparation for the field theory limit, $\alpha' \rightarrow 0$. The result takes the form

$$A_2^{(1)} = \frac{N}{2} \frac{g^2}{(4\pi)^2} \delta_{ab} (4\pi\mu^2 \cdot 2\alpha')^{2-d/2} (\varepsilon_1 \cdot \varepsilon_2 p_1 \cdot p_2 - \varepsilon_1 \cdot p_2 \varepsilon_2 \cdot p_1) R(p_1 \cdot p_2) . \quad (11)$$

Here we wrote $g_d = g\mu^\epsilon$ in accordance with field theory conventions, and we have chosen not to enforce transversality and momentum conservation just yet, to display the correspondence with the field theory off-shell amplitude. The relevant factor is the integral $R(p_1 \cdot p_2)$, defined by

$$R(s) \equiv \int_0^\infty d\tau \int_0^\tau d\nu \left[e^{2\tau} (\tau)^{-d/2} \prod_{n=1}^\infty \left(1 - e^{-2n\tau}\right)^{2-d} e^{2\alpha's G(\nu)} (\partial_\nu G(\nu))^2 \right] . \quad (12)$$

Eq. (12) is the starting point to take the field theory limit, which in these coordinates corresponds to the limit $k = e^{-2\tau} \rightarrow 0$, or $\tau \rightarrow \infty$, as in [6, 17]. In this limit the Green's function behaves as

$$G(\nu) = -\frac{\nu^2}{\tau} + \log(2 \sinh(\nu)) - 4e^{-2\tau} \sinh^2(\nu) + O(e^{-4\tau}) , \quad (13)$$

and it is convenient to change variables to $\hat{\nu} = \nu/\tau$. Substituting Eq. (13) into Eq. (12), and keeping only the terms that are finite in the limit $k = 0$ (the divergent terms correspond to the infrared divergence associated with the tachyon), we find

$$R(s) = \int_0^\infty d\tau \int_0^1 d\hat{\nu} \tau^{1-d/2} e^{2\alpha's(\hat{\nu}-\hat{\nu}^2)\tau} [(1-2\hat{\nu})^2(d-2) - 8] . \quad (14)$$

The integrals are now elementary, and the answer is, reverting to the dimensional continuation parameter ϵ ,

$$R(s) = -2 \Gamma(\epsilon) (-2\alpha' s)^{-\epsilon} \frac{11 - 7\epsilon}{3 - 2\epsilon} B(1 - \epsilon, 1 - \epsilon) . \quad (15)$$

Putting everything together we find that the two-gluon one-loop amplitude is

$$\begin{aligned} A_2^{(1)} &= -N\delta_{ab} \frac{g^2}{(4\pi)^2} \left(\frac{4\pi\mu^2}{-p_1 \cdot p_2} \right)^\epsilon (\varepsilon_1 \cdot \varepsilon_2 \ p_1 \cdot p_2 - \varepsilon_1 \cdot p_2 \ \varepsilon_2 \cdot p_1) \times \\ &\times \Gamma(\epsilon) \frac{11 - 7\epsilon}{3 - 2\epsilon} B(1 - \epsilon, 1 - \epsilon) . \end{aligned} \quad (16)$$

This result is exactly equal, including the normalization and to all orders in ϵ , to the result one obtains calculating the gluon vacuum polarization in field theory, using the background field method and the background field Feynman gauge, with Feynman rules given for example in Ref. [18] (the sign of $p_1 \cdot p_2$ is due to the fact that we are working with the metric of Ref. [12]). One is lead to the conclusion (which we will verify in the last section) that this is the gauge to which our prescription leads, as might have been expected from the on-shell analysis of Ref. [19]. In particular, the divergence in Eq. (16) must be cancelled by a wave function renormalization, and using the background field method Ward identity $Z_g = Z_A^{-1/2}$ and minimal subtraction one recovers the correct Yang-Mills β function.

4 Bosonic string in an external background

In this section we will follow a different approach, which also leads to a correct determination of the Yang-Mills β function. We will study an open bosonic string in interaction with a slowly varying external non-abelian field, and we will extract the one-loop contribution to the two point function. In the one-loop effective action this corresponds to a renormalization of the gauge field classical action, *i.e.* to a wave function renormalization.

Let us consider the partition function of an open bosonic string interacting with an external non-abelian $SU(N)$ background, pointing in a specific direction in colour space, and let us denote it by $Z(F_{\mu\nu})$. We write the string coordinate $X^\mu(z)$ as

$$X^\mu(z) = \frac{x^\mu}{\sqrt{2\alpha'}} + \xi^\mu(z) , \quad (17)$$

so that X^μ and ξ^μ are dimensionless, while the zero mode x^μ has dimensions of length; then the planar contribution to $Z(F_{\mu\nu})$ is given by

$$\begin{aligned} Z_{pl}(F_{\mu\nu}) &= \sum_{h=0}^{\infty} N^h g_s^{2h-2} \int \frac{d^d x}{(2\alpha')^{d/2}} \int_h D\xi Dg e^{-S_0(\xi, g; h)} \times \\ &\times Tr \left[P_z \exp \left(-i\alpha' g_d F_{\mu\nu}(x) \int_h dz \partial_z \xi^\mu(z) \xi^\nu(z) \right) \right] . \end{aligned} \quad (18)$$

Here the symbol P_z reminds us that in the open string the z variables along the world-sheet boundary are ordered; g_s is the dimensionless string coupling constant, related to g_d through

$$g_s = \frac{g_d}{\sqrt{2}} (2\alpha')^{1-d/4} ; \quad (19)$$

$F_{\mu\nu}$ is a matrix proportional to one of the $SU(N)$ generators λ_a (normalized by $\text{Tr}(\lambda_a \lambda_b) = \frac{1}{2}\delta_{ab}$), and we take it to be independent of ξ^μ , and slowly varying as a function of the zero mode x^μ ; finally, the classical action on a genus h manifold with metric $g_{\alpha\beta}$ is

$$S_0(\xi, g; h) = \frac{1}{2\pi} \int_h d^2 z \sqrt{g} g^{\alpha\beta} \partial_\alpha \xi \cdot \partial_\beta \xi . \quad (20)$$

We are interested in computing the contribution to Eq. (18) which is quadratic in the field strength, and which shall be denoted by $Z_2(F_{\mu\nu})$. It is given by

$$\begin{aligned} Z_2(F_{\mu\nu}) &= \sum_{h=0}^{\infty} N^h g_s^{2h} \left[-\frac{1}{4} \int d^d x F_{\mu\nu}^a(x) F_{\rho\sigma}^a(x) \right] \times \\ &\quad \times \frac{1}{2} \int dz dw \langle \partial_z \xi^\mu(z) \xi^\nu(z) \partial_w \xi^\rho(w) \xi^\sigma(w) \rangle_h , \end{aligned} \quad (21)$$

where a is a fixed index, and the correlator is defined by the functional integral in Eq. (18), formally continued to arbitrary d . The contribution from the disk ($h = 0$) is just the classical gauge action for the background field,

$$Z_2^{(0)} = -\frac{1}{4} \int d^d x F_{\mu\nu}^a F_{\mu\nu}^a . \quad (22)$$

The contribution from the annulus ($h = 1$) has the same form, but it is multiplied by a non-trivial integral,

$$Z_2^{(1)} = -\frac{1}{4} \int d^d x F_{\mu\nu}^a F_{\mu\nu}^a \hat{Z}_2 , \quad (23)$$

where

$$\hat{Z}_2 = \frac{N}{2} \frac{g_d^2}{(2\pi)^d} (2\alpha')^{2-d/2} \int \frac{[dm]_1}{z_2} \lim_{z_1 \rightarrow 1} \left[\partial_{z_1} \mathcal{G}(z_1, z_2) - \frac{1}{2z_1} \right]^2 , \quad (24)$$

while $\mathcal{G}(z_1, z_2)$ is the bosonic Green's function given in Eq. (9), and $[dm]_1$ is the measure given in Eq. (7).

Introducing the variables ν and τ defined below Eq. (8), we can rewrite the expression in Eq. (24) as

$$\begin{aligned} \hat{Z}_2 &= \frac{N}{2} \frac{g_d^2}{(4\pi)^{d/2}} (2\alpha')^{2-d/2} \int_0^\infty d\tau e^{2\tau} (\tau)^{-d/2} \prod_{n=1}^{\infty} \left(1 - e^{-2n\tau} \right)^{2-d} \int_0^\tau d\nu [\partial_\nu G(\nu)]^2 \\ &= \frac{N}{2} \frac{g^2}{(4\pi)^2} (4\pi\mu^2 \cdot 2\alpha')^{2-d/2} R(0) , \end{aligned} \quad (25)$$

where $R(s)$ was defined in Eq. (12).

We see that we recover precisely the expression of Eq. (11), but with the infrared cutoff $p_1 \cdot p_2 = -m^2$ removed inside the integral. The reason is that the calculation leading to Eq. (11) can also be understood as a background field calculation, but with the background represented by a plane wave vector potential. Here instead we have used a vector potential linear in x , leading to a constant field strength. The present calculation is thus the long-wavelength approximation of the previous one, where we have retained only terms linear in the momentum p , so that the exponential factor in $R(p_1 \cdot p_2)$ has been discarded.

Clearly, the elimination of the exponential term, which acted as an infrared cutoff, will generate contributions from the massless states, that will survive in the field theory limit, and will be infrared divergent in the limit $d \rightarrow 4$, as expected from field theory.

We can see this explicitly if we repeat the procedure leading to Eq. (14), discarding as before the tachyonic contributions.

Performing the integration over the puncture we get

$$\hat{Z}_2 = \frac{N}{2} \frac{g_d^2}{(4\pi)^{d/2}} \frac{d-26}{3} (2\alpha')^{2-d/2} \int_0^\infty d\tau \tau^{1-d/2} . \quad (26)$$

In $d > 4$ the integral over τ is ultraviolet divergent. It can be regularized by introducing a cutoff $1/(2\alpha'\Lambda^2)$ at the lower limit of integration, corresponding to a proper time cutoff in field theory. Then we get

$$\hat{Z}_2 = N \frac{g_d^2}{(4\pi)^{d/2}} \frac{d-26}{3} \frac{\Lambda^{d-4}}{d-4} , \quad (27)$$

which agrees with the result obtained in Refs. [1, 20], and gives a logarithmic ultraviolet divergence as $d \rightarrow 4$, as expected. We have also verified that, if one calculates the two-gluon amplitude with the background field method and in the background Feynman gauge, as suggested in the previous section, but regularizing the loop integral by means of a proper-time cutoff, rather than dimensional regularization, the term proportional to Λ^{d-4} is precisely given by Eq. (27). This is not a surprise, but it strengthens the connection between string theory and the background field method.

When $d = 4$, the ultraviolet divergence in Eq. (26) can still be regularized as in Eq. (27), or with dimensional regularization, but the integral develops an infrared divergence, as is expected from field theory and from the analogous calculations performed for the closed string [3]. One can then reintroduce an infrared cutoff in exponential form, in analogy with the calculation leading to Eq. (15). Multiplying the integrand in Eq. (26) with a factor $e^{-2\alpha'm^2\tau}$ we get

$$\hat{Z}_2 = \frac{N}{2} \frac{g_d^2}{(4\pi)^{d/2}} \frac{d-26}{3} m^{d-4} \Gamma(2-d/2) . \quad (28)$$

Once again, expanding around four dimensions ($d = 4 - 2\epsilon$) and using $g_d = g\mu^\epsilon$, we get the pole contribution

$$\hat{Z}_2 = -N \frac{g^2}{(4\pi)^2} \frac{11}{3} \frac{1}{\epsilon} , \quad (29)$$

thus reproducing the results of the previous section and the correct Yang-Mills β function.

If we had chosen from the beginning to compactify some of the string coordinates, Eq. (29) would have received contributions also from the adjoint scalars associated with the compactified dimensions. In this case Eq. (29) generalizes to

$$\hat{Z}_2 = -N \frac{g^2}{(4\pi)^2} \left[\frac{11}{3} - \frac{N_s}{6} \right] \frac{1}{\epsilon} , \quad (30)$$

where N_s is the number of compactified dimensions, and the number of scalars, so that we recover the well known result stating that the β function of four-dimensional Yang-Mills theory coupled with 22 adjoint scalars vanishes.

5 Consistency checks and conclusions

In section 2 we have shown that when the two-gluon amplitude at one loop is suitably extended off shell, it leads in the field theory limit, Eq. (16), to the wave function renormalization

$$Z_A = 1 + N \frac{g^2}{(4\pi)^2} \frac{11}{3} \frac{1}{\epsilon} , \quad (31)$$

which agrees with the result of the background field calculation performed in section 3 and given by Eq. (29). This result suggests that our prescription to continue the gluon amplitudes off-shell gives the renormalization constants of the background field method. To verify this statement we have to check the background field Ward identity

$$Z_g = Z_A^{-1/2} , \quad (32)$$

where as usual $Z_g = Z_3 Z_A^{-3/2}$. This requires that we compute the three-gluon amplitude $A_3^{(1)}$, continue it off-shell according to the same prescription used for the two-point amplitude, and then extract from it the ultraviolet divergent contribution in the field theory limit. The latter, according to Eq. (2), must be given by

$$A_{3,\text{div}}^{(1)} = (Z_3^{-1} Z_A^3 - 1) A_3^{(0)} , \quad (33)$$

so that we can extract from it the vertex renormalization constant Z_3 , and thus Z_g . Further, we must verify that our prescription corresponds to a consistent one-loop renormalization prescription for the gauge theory, respecting gauge invariance. To do this we have to compute the ultraviolet divergence of the four-point amplitude,

and verify that the Ward identity $Z_4 = Z_3^2 Z_A^{-1}$ is also satisfied, so that the renormalized coupling appearing in the four-point vertex is in fact the square of the renormalized three-gluon coupling.

The details of this calculation are left to a subsequent publication [13]; here we simply give a brief summary of our results.

The three-gluon amplitude $A_3^{(1)}$ is an integral over the modulus of the annulus and three cyclically ordered punctures ν_1 , ν_2 and ν_3 , one of which, say ν_1 , is actually fixed to zero. According to [6], the terms that are divergent in the field theory limit are those which arise from “pinching” together two of the three punctures. Because of the cyclic order, only two pinchings are possible, namely

$$\nu_2 \rightarrow \nu_1 = 0 \quad \text{and} \quad \nu_3 \rightarrow \nu_2 . \quad (34)$$

Each one of these pinchings isolates the loop on an external leg and gives rise to an expression which has poles associated to all possible string states that are exchanged in the intermediate channel. The only term in this expansion that survives in the field theory limit is the one corresponding to the exchange of an intermediate gluon. The coefficient of this term contains ratios of vanishing momentum invariants, which can be given an unambiguous meaning using the off-shell prescription of Eq. (4). We explicitly verified [13] that, when the pinchings are performed and the field theory limit is taken, the divergent part of three-gluon amplitude at one loop can be written as

$$A_{3,\text{div}}^{(1)} = (Z_A^2 - 1) A_3^{(0)} , \quad (35)$$

which implies the background field Ward identity $Z_3 = Z_A$, and thus Eq. (32). Similarly, we verified from the four-gluon amplitude at one loop that $Z_4 = Z_A$, so that also the Ward identity $Z_4 = Z_3^2 Z_A^{-1}$ is satisfied, and the theory can be renormalized at one-loop respecting gauge invariance.

The lessons we have learnt can be summarized as follows. Existing calculations of gauge-theory renormalization constants from string theory are all strictly related to the background field method. The calculation of the string effective action in a slowly varying background gauge field can be seen as the long wavelength limit of the same calculation performed with a plane-wave background, and this in turn coincides with the amplitude for the scattering of the appropriate number of gluons. Gluon scattering amplitudes in the field theory limit can be consistently continued off shell, and to arbitrary space-time dimension, recovering the background field Feynman gauge in dimensional regularization. The correspondence persists if one uses a proper-time cutoff instead of dimensional continuation. We hope that the strong simplifications of the calculations due to the fact that we are dealing with such a simple string theory will make it possible to extend the present results to higher loops.

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